UDC 519

ON THE TYPICAL STRUCTURE OF THE CONDITIONAL SCALE-FREE RANDOM GRAPH

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This paper focuses on the typical limit structure of the scale-free random graph under the condition that a known number of edges and number of vertices tend to infinity.

Keywords: random graph, power-law distribution, limit behavior.

In recent years growing attention of researchers has been attracted by the problem of modeling complex networks, for instance the Internet and social networks. Quite a number of books and papers are devoted to the applications of the random graph theory for the study of networks topology (see e.g. [1-4] and references therein). The present paper focuses on the model introduced in [5, 6] and called there the power-law random graph or scale-free random graph (see [4]), which means that degrees of vertices are independent of the size of the graph.

Let the number of vertices in the graph be equal to $N+1$ and the vertices have numbers $0,1,\ldots,N$. The degrees of vertices $1,\ldots,N$ are the independent identically distributed random variables $\xi_1,\ldots,\xi_N$ such that

$p_k = \mathbf{P}\{\xi_1 = k\} = k^{-\tau} - (k+1)^{-\tau}, k = 1,2,\ldots, \tau > 0.$ (1)

In [1] (see also [3, 4]) it was shown that in practice the typical value of parameter τ belongs to the interval (1,2). For the sake of simplicity in [2] the notion "stub" was introduced, defined as an edge having only one vertex on its end, the second end being free at the beginning. All stubs are different (enumerated). The graph is constructed by joining each stub to another equiprobably to form edges. Since the sum of degrees must be even we introduce a new random variable ξ_0 which corresponds to the vertex with number 0 and is equal to 1 if the sum $\zeta_N = \xi_1 + \ldots + \xi_N$ is odd, zero otherwise.

The results describing the limit behavior (as $N \to \infty$) of many characteristics of random graphs were obtained in [2-4, 7]. It is shown that the random graph has a unique giant component, i.e. is a connected component with size proportional to N. In [2] it was found that vertices with big degrees form the core of the giant component. This core has a hierarchical structure and can be thought to be decomposed into layers surrounding the vertex with the largest degree such that vertices are always directly connected to some vertex of the next higher layer. These results were employed to find the estimation of the mean size of the giant component and the upper bound of the length of the path between two randomly chosen vertices. The limit distribution of the giant component size was obtained in [7].

A sum of the vertex degrees ζ_N is a random variable. In practice we can sometimes estimate the number of links in a network, so it is interesting to study the set of graphs with a given number of edges. We consider the subset of the power-law random graphs under the conditions that $\tau \in (1,2)$ and $\zeta_N = n$. Investigation of such random graphs began in [8] where limit distributions of the maximum vertex degree and of the number of vertices with given degree were obtained. The technique of proof of these theorems is based on the generalized allocation scheme suggested by V.F. Kolchin [9].

The aim of these notes is to identify the zone of parameters behavior when the asymptotic structure of conditional random graph is the same structure as the random graph without the condition $\zeta_N = n.$

We say that a sequence of events $A_k, k = 1,2,\ldots$ occurs with high probability (w.h.p.) when $\mathbf{P}\{A_k\} \to 1$ as $k \to \infty.$ Let

$q_k = (k+1)p_{k+1}\zeta^{-1}(\tau),$ (2)

where $k = 0,1,2,\ldots, p_k$ are defined by (1), $\zeta(\tau)$ is the Rimann's zeta-function. We denote by $\varphi(z)$ the characteristic function of the distribution (1). We set also

$\alpha = \lceil \ln\ln N/(-\ln(\tau-1)) \rceil,\ \beta = \lceil \exp\{2w/(2-\tau)\} \rceil,$

where $\lceil x \rceil$ denotes the least natural number such that $\lceil x \rceil \geq x,\quad w = w(N)$ is an increasing function and $w/\ln\ln\ln N \to 0$, $w/\ln\ln\ln\ln N \to \infty$. The next assertion is valid.

Theorem. Let $N, n \to \infty$ in such a way that $n - \zeta(\tau)N = O(N^{1/\tau}).$ Then the conditional random graph has a connected component with size V which satisfies the next conditions: $\mathbf{E}V/N \to 1-\varphi(q),$ where q is the extinction probability of the Galton-Watson branching process with offspring distribution (2) and the diameter of this component w.h.p. is $2(\alpha+\beta)$ at most.

Proof of this theorem is based on the use of the results obtained in [2] but we also widly use the theorems about limit distributions of vertex degrees under the condition $\zeta_N = n$ [8].

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О ТИПИЧНОЙ СТРУКТУРЕ УСЛОВНОГО БЕЗМАСШТАБНОГО СЛУЧАЙНОГО ГРАФА

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В статье рассматривается типичная предельная структура безмасштабного случайного графа при условии, что известное число ребер и число вершин стремятся к бесконечности.

Ключевые слова: случайный граф, степенной закон распределения, предельное поведение.