The Fourth Main Boundary Value Problem of Dynamics of Thermo-resiliency’s Momentum Theory

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Abstract
In the paper is presented the fourth main boundary value problem of Dynamics of Thermo-resiliency’s Momentum theory. The problem states to find in the cylinder $D_l$ the regular solution of the system:

$$M(\partial_t)U - \nu \chi \theta - \chi^0 \frac{\partial^2 u}{\partial t^2} = \mathcal{H}, \quad \Delta \theta - \frac{1}{\delta} \frac{\partial \theta}{\partial t} - \eta \frac{\partial}{\partial t} \operatorname{div} u = \mathcal{H}_T,$$

which satisfies the initial conditions:

$$\forall x \in D: \lim_{t \to 0} U(x, t) = \varphi^{(0)}(x), \lim_{t \to 0} \theta(x, t) = \varphi_T^{(0)}(x), \quad \lim_{t \to 0} \frac{\partial u(x, t)}{\partial t} = \varphi^{(1)}(x)$$

and the boundary conditions:

$$\forall (x, t) \in S_l: \lim_{\exists x \to y \in S} PU = f, \quad \lim_{\exists x \to y \in S} \{\theta\}^+_S = f_T.$$

The uniqueness theorem of the solution is proved for this problem.

Keywords: the main boundary value problem; initial conditions; boundary conditions; the uniqueness theorem of the solution.

Introduction
Let $D$ be a finite or infinite three-dimensional space with the compact boundary $S$ from the class $A_2(\alpha), (\alpha > 0)$.

Denote by $D_l$ and $S_l$ cylinders $D_l = D \times l$, $S_l = S \times l$, respectively, where $l = [0, \infty)$. 
In the problems of Dynamics of Thermo-resiliency’s Momentum theory any point of environment is characterized by seven quantities: a movement vector \(-u = (u_1, u_2, u_3)\), a rotation vector \(-\omega = (\omega_1, \omega_2, \omega_3)\) and a temperature deviation \(-\theta\).

The main equations of the Thermo-resiliency’s Momentum theory can be written in a matrix form as follows \[1, \; 2\]:

\[
M(\partial_x) U - \nu \chi \theta - \chi^0 \frac{\partial^2 U}{\partial t^2} = \mathcal{H}, \quad \Delta \theta - \frac{1}{\eta} \frac{\partial}{\partial x} - \eta \frac{\partial}{\partial x} \frac{\partial v}{\partial t} u = \mathcal{H}_t, \tag{1}
\]

where \(M(\partial_x)\) is a matrix differential operator of the Momentum Resilience theory \[3\] and \(\chi = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, 0, 0, 0)\), \(\chi^0 = \|\chi \|_{\infty}^0\) for \(i = 1, 2, 3\), \(\chi^0 = \zeta\) for \(i = 4, 5, 6\), \(\chi^0 = 0\) for \(i \neq j\), \(\mathcal{H} = (-\rho F, -\rho Y), \mathcal{H}_t = -\frac{1}{\partial} Q, U = (u, \omega)\).

Let \(\phi(i) = \left(\phi_1(i), \phi_2(i)\right)\) for \(i = 0, 1\), where \(\phi_1(i) = (\phi_1^k(i), \phi_2^k(i), \phi_3^k(i))\) for \(k = 1, 2\) and \(\phi_7(i)\) for \(i = 0, 1\) be functions given in the area \(\mathcal{D}\), while \(f = (f^1, f^2), f^i = (f_1^i, f_2^i, f_3^i)\) for \(i = 1, 2\) and \(f_7\) are functions given on \(S_t\).

**Definition**

Vector-function \(U = (u, \omega, \theta)\) is called as regular in the area \(D_t^+\) if \(U \in C^1(\mathcal{D}_t^+) \cap C^2(D_t^+)\) for \(\forall t \in I\) and \(B(\partial_x, \partial t)U\) is integrable in the area \(D^+\).

Analogously, vector-function \(U = (u, \omega, \theta)\) is called as regular in the area \(D_t^-\) if \(U \in C^1(\mathcal{D}_t^-) \cap C^2(D_t^-)\) for \(\forall t \in I\) and \(B(\partial_x, \partial t)U\) is integrable in the area \(D^- \cap M(0, \delta)\) for any number \(\delta > 0\) and

\[
|U(x, t)| \leq \frac{c(t)}{1+|x|^2} \left|\frac{\partial U(x, t)}{\partial t}\right| \leq \frac{c(t)}{1+|x|^2} \left|\frac{\partial U(x, t)}{\partial x}\right| \leq \frac{c(t)}{1+|x|^2}, \tag{2}
\]

where \(B(\partial_x, \partial t)\) is an operator standing on the left side of the system (1) and is written in the form of a matrix differential operator.

In the paper is studied the following problem of Dynamics of Thermo-resiliency’s Momentum theory: to find in the cylinder \(D_t\) the regular solution of the system (1) which satisfies the initial conditions:

\[
\forall x \in D: \lim_{t \to 0} U(x, t) = \phi^{(0)}(x), \lim_{t \to 0} \theta(x, t) = \phi_7^{(0)}(x), \lim_{t \to 0} \frac{\partial U(x, t)}{\partial t} = \phi^{(1)}(x)
\]

and the boundary conditions:

\[
\forall (x, t) \in S_t: \lim_{\partial S_t = y \in \mathcal{S}} PU = f, \lim_{\partial S_t = y \in \mathcal{S}} (\theta_7)^{+} = f_7.
\]

Here, \(P = P(\partial_x, n)\) is an operator of thermo-momentary voltage:

\[
P(\partial_x, n)U = T(\partial_x, n)U - ve\theta,
\]

where \(T(\partial_x, n)\) is an operator of momentary voltage \[3\], \(e = (n_1, n_2, n_3, 0, 0, 0), U = (u, \omega)\) and \(n(n_1, n_2, n_3)\) is a normal of the surface \(S\).

**The main result**

The following uniqueness theorem is true:

**Theorem.** In the cylinder \(D_t^+\) the regular solution of the homogenous problem, corresponding to the above stated problem, is identical to 0.

**The proof of the theorem.** Let \(U = (U, \theta)\) be a regular solution in \(D_t^+\) of the homogenous equation corresponding to (1). Then, the following formula is true:

\[
\frac{\partial}{\partial t} \int_D \left\{ \frac{1}{2} \sum_{i=1}^{6} \chi_i^0 \left|\frac{\partial U}{\partial t}\right|^2 + \frac{1}{2} E(U, U) + \frac{\nu}{2\eta} |\theta|^2 \right\} dx + \frac{\nu}{\eta} \int_D |grad \theta|^2 dx = \int_S \left\{ \frac{\partial U}{\partial t} PU - \frac{\nu}{\eta} \theta \frac{\partial \theta}{\partial n} \right\} dS,
\tag{3}
\]

where \(E(U, U)\) is a positively defined form \[3\].

For the regular solution of the homogenous problem in \(D_t^+\) the right side of (3) is equal to 0. Hence, the left side of it is also equal to 0, from which follows that \(U = 0, \theta = 0\). So, \(U = 0\).

Now, let \(U = (U, \theta)\) be a regular solution of the homogenous problem in \(D_t^+\), corresponding to the system (1). We can write (3) for \(D^- \cap M(0, z)\) as follows:
\[
\frac{\partial}{\partial t} \int_{D \cap M(0, z)} \left\{ \frac{1}{2} \sum_{i=1}^{6} x_{ii}^0 \left( \frac{\partial U_i}{\partial t} \right)^2 + \frac{1}{2} E(U, U) + \frac{\nu}{2\eta} |\theta|^2 \right\} dx + \\
+ \frac{\nu}{\eta} \int_{D \cap M(0, z)} |\text{grad } \theta|^2 dx = \int_{C(0, z)} \left\{ \frac{\partial U}{\partial t} PU + \frac{\nu}{\eta} \frac{\partial \theta}{\partial n} \right\} dS,
\]
where \( z \) is a sufficiently large number.

Considering the conditions (2) and taking the limit of the above equation as \( z \to \infty \), we get that

\[
\frac{\partial}{\partial t} \int_{D} \left\{ \frac{1}{2} \sum_{i=1}^{6} x_{ii}^0 \left( \frac{\partial U_i}{\partial t} \right)^2 + \frac{1}{2} E(U, U) + \frac{\nu}{2\eta} |\theta|^2 \right\} dx + \frac{\nu}{\eta} \int_{\partial D} |\text{grad } \theta|^2 dx = 0,
\]
from which, using the homogeneity of the initial conditions, we have:

\[ U = 0. \]

Thus, the theorem is proved.

**Conclusion**

The main task was to prove the uniqueness theorem of the solution of the fourth main boundary value problem of Dynamics of Thermo-resiliency’s Momentum theory. In the cylinder \( D_l \) was found the regular solution of the system:

\[
M(\partial_x)U - \nu x \theta - x^0 \frac{\partial^2 u}{\partial t^2} = \mathcal{H}, \quad \Delta \theta - \frac{1}{\theta} \frac{x \theta}{\partial t} - \eta \frac{\partial}{\partial t} \text{ div } u = \mathcal{H}_\gamma,
\]
which satisfies the following initial and boundary conditions:

\[
\forall x \in D: \quad \lim_{t \to 0} U(x, t) = \phi^{(0)}(x), \quad \lim_{t \to 0} \theta(x, t) = \phi^{(1)}_\gamma \theta(x), \quad \lim_{t \to 0} \frac{\partial u(x, t)}{\partial t} = \phi^{(0)}(x);
\]

\[
\forall (x, t) \in S_l: \quad \lim_{D \ni x \to y \in S} PU = f, \quad \lim_{D \ni x \to y \in S} \{\theta\}^+_S = f_\gamma.
\]

**References:**